ON THE EXISTENCE OF REGULAR JACOBI STRUCTURES

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Abstract. We prove h-principle for locally conformal symplectic foliations and contact foliations on open manifolds. We interpret the result on h principle of contact foliations in terms of the regular Jacobi structures.

Key words: Contact foliations, locally conformal symplectic foliations, regular Jacobi structures.

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1. Introduction

It is a classical result due to Gromov which says that if an *open* manifold admits a non-degenrate 2-form then it must also admit a symplectic form. A foliated version of this result was proved by Bertelson in [1] under some open-ness assumption on the foliation.

Theorem 1.1. *Any 2-form on a foliated manifold $(M, \mathcal{F})$ which is non-degenerate on the leaves is homotopic through the space of such 2-forms to a leafwise symplectic form, provided the foliation $\mathcal{F}$ satisfies some 'openness' condition.*

The theorem follows as a direct consequence of a general result in h-principle proved in [1], in which the author has also produced several examples to show that the theorem breaks down without the open-ness condition on the foliation.

Symplectic foliations $(M, \mathcal{F})$ are closely related to regular Poisson structures on M. Recall that a Poisson structure π is a bivector field satisfying the condition $[\pi, \pi] = 0$, where the bracket denotes the Schouten bracket of multivector fields. The bivector field π induces a vector bundle morphism $\pi^\# : T^*M \to TM$ by $\pi^\#(\alpha)(\beta) = \pi(\alpha, \beta)$ for all $\alpha, \beta \in T^*_xM$, $x \in M$. The characteristic distribution $\mathcal{D} =$ Image $\pi^\#$, in general, is a singular distribution which however integrates to a foliation. The restrictions of the Poisson structure to the leaves define symplectic forms on the leaves in a canonical way. Hence, the characteristic foliation is a (singular) symplectic foliation. In fact, the restriction of the Poisson structure to a leaf of the foliation has the maximum rank and so it canonically defines a symplectic form on the leaf. A Poisson bivector field π is said to be regular if the rank of $\pi^\#$ is constant. In this case the characteristic foliation is a regular symplectic foliation on M. On the other hand, given a regular symplectic foliation $\mathcal{F}$ on M one can associate a Poisson bivector field π having $\mathcal{F}$ as its characteristic foliation. Theorem 1.1 thus can be stated as follows:

If π_0 is a regular bivector field whose characteristic distribution $\mathcal{D} = Im\ {\pi_0}^\#$ integrates to a foliation satisfying some open-ness condition, then it can be homotoped through regular bivector fields π_t to a Poisson bivector field π_1 having the

same characteristic distribution $\mathcal{D}$. Moreover, the underlying distribution remains constant throughout the deformation π_t.

In fact, in trying to keep the same underlying foliation, Bertelson had to impose the additional condition on $\mathcal{F}$. In a recent article, Fernandes and Frejlich [5] have shown that if we allow the underlying foliation to vary then a regular bivector field π_0 whose characteristic distribution is integrable can be homotoped to a Poisson bivector field, provided the manifold is open. In general, an arbitrary distribution need not be integrable. Taking into account Haefliger's result [8] Fernandes and Frejlich have proved the following.

Theorem 1.2. *[5] Every regular bivector field π_0 on an open manifold can be homotoped to a Poisson bivector provided the underlying distribution of π_0 is homotopic to an integrable one.*

In [5], the authors further comment that there should be analogues of this result for foliated locally conformal symplectic manifolds, foliated contact manifolds or more generally for Jacobi manifolds. Motivated by this, we prove the following results.

Theorem 1.3. *Let M^{2n+q} be an open manifold with a codimension q foliation $\mathcal{F}_0$ and a 2-form ω_0 which is non-degenerate on the leaves of $\mathcal{F}_0$. Fix a deRham cohomology class $\xi \in H^1_{dR}(M;\mathbb{R})$. Then there exists a homotopy $(\mathcal{F}_t, \omega_t)$ and a closed 1-form θ representing ξ such that*

(1) *ω_t is $\mathcal{F}_t$-leafwise non-degenerate*
(2) *ω_1 satisfies the relation $d\omega_1 - \theta \wedge \omega_1 = 0$.*

In particular, ω_1 is $\mathcal{F}_1$-leafwise locally conformal symplectic form with the leafwise Lee class $\xi_{\mathcal{F}}$ in the foliated cohomology group $H^2(M, \mathcal{F}_1)$ determined by ξ.

As an immediate corollary we can deduce that any open manifold M admits a locally conformal symplectic form ω with a given Lee class ξ (see prelimaries) provided that there exists a non-degenerate 2-form on M. Further, when $\xi = 0$ we get a globally conformal symplectic form and hence a symplectic form on the manifold. Thus we can recover Gromov's theorem on the existence of Symplectic form on open manifold.

Theorem 1.4. *Let $M^{(2n+1)+q}$ be an open manifold and $\mathcal{F}_0$ a codimension q foliation on M. Let $(\theta_0, \omega_0) \in \Omega^1(M) \times \Omega^2(M)$ be such that the restrictions of $\theta_0 \wedge (\omega_0)^n$ to the leaves of $\mathcal{F}$ are nowhere vanishing. Then there exists a homotopy $(\mathcal{F}_t, \theta_t, \omega_t)$ such that $\theta_t \wedge (\omega_t)^n \neq 0$ on the leaves of $\mathcal{F}_t$ and $\omega_1 = d\theta_1$. In particular, θ_1 is a leafwise contact form on $(M, \mathcal{F})$.*

As a corollary we can recover Gromov's result on the existence of a contact form on an open manifold [3].

Finally to link the results with Jacobi structures we would like to mention that locally conformal symplectic foliations and contact foliations are nothing but the characteristic foliations associated with regular Jacobi structures (see Section 4). Thus the last two results throw some light on the existence of regular Jacobi structures on open manifolds.

We organise the paper as follows. The proofs of Theorems 1.3, 1.4 are given in Sections 2 and 3 respectively. In section 3 we interpret Theorem 1.4 in terms of

regular Jacobi structure on a manifold. In Appendix 1 (Section 4) we recall some basic theory of Jacobi manifolds. We briefly discuss the Holonomic Approximation Theorem in Appendix 2 (Section 5) as it plays an important role in the proof.

2. The foliated l.c.s case

In this section we prove that an open manifold together with a regular foliation and a leafwise non-degenerate 2-form can be homotoped through such pairs to a regular foliation with a leafwise locally conformal symplectic form having prescribed Lee class.

Lemma 2.1. *Let M^n be a given manifold equiped with a 1-form θ. Then there exists a map $D_\theta : E^{(1)} = (\wedge^1 M)^{(1)} \to \wedge^2 M$ satisfying $D_\theta \circ j^1\alpha = d_\theta\alpha$ and is such that the following diagram is commutative.*

$$\begin{array}{ccc} E^{(1)} & \xrightarrow{D_\theta} & \wedge^2(M) \\ \downarrow & \circlearrowleft & \downarrow \\ M & \xrightarrow{id_M} & M \end{array}$$

Further, the fibres of D_θ are affine subspaces.

Proof. Choose local coordinates $(x^1, ..., x^n)$ around $x_0 \in M$. So in this local coordinates α and θ takes the form $\alpha = \Sigma_{i=1}^n \alpha_i dx^i$, $\theta = \Sigma_{i=1}^n \theta_i dx^i$ which gives $j^1_{x_0}\alpha = (dx^1, ..., dx^n)(a_1, ..., a_n)^T + (dx^1, ..., dx^n)A(x^1 - x^1_0, ..., x^n - x^n_0)^T$, where $a_i = \alpha_i(x_0)$, $A = (a_{ij})_{n\times n} = (\frac{\partial \alpha_i}{\partial x^j}(x_0))_{n\times n}$. So $j^1_{x_0}\alpha$ can be thought of as an element of $\mathbb{R}^{(n+n^2)}$, i.e $j^1_{x_0}\alpha = (a_i, a_{ij}) \in \mathbb{R}^{(n+n^2)}$. Now we define D_θ as

$$D_\theta(j^1_{x_0}\alpha) = \Sigma_{i<j}[(a_{ij} - a_{ji}) + (\theta_i(x_0)a_j - a_i\theta_j(x_0))]dx^i \wedge dx^j$$

so that $D_\theta(j^1_{x_0}\alpha) = [d\alpha - \theta \wedge \alpha]_{x_0} = d_\theta(\alpha)_{x_0}$.

Observe that D_θ is a vector bundle epimorphism. Indeed, given $b_{ij}, 1 \le i < j \le n$ the following system of linear equations

$$(a_{ij} - a_{ji}) + (a_i\theta_j(x_0) - a_j\theta_i(x_0)) = b_{ij}$$

is clearly solvable by $a_i = 0$, $a_{ij} = -a_{ji} = \frac{b_{ij}}{2}$. Moreover, the fibres of D_θ are affine subspaces of dimension $\frac{n^2+3n}{2}$. □

Remark: In the above θ has not been taken to be a closed form. Closedness of θ is only needed to make d_θ a coboundary operator, where $d_\theta = d - \theta\wedge$.

Corollary 2.2. *Every section $\omega : M \to \wedge^2 M$ can be lifted up to a section $F_\omega : M \to (\wedge^1 M)^{(1)}$ such that $D_\theta F_\omega = \omega$, i.e*

$$\begin{array}{ccc} E^{(1)} & \xrightarrow{D_\theta} & \wedge^2(M) \\ F_\omega\uparrow\downarrow & \circlearrowleft & \downarrow\uparrow\omega \\ M & \xrightarrow{id_M} & M \end{array}$$

Moreover, any two such lifts of a given section $\omega : M \to \wedge^2 M$ is canonically homotopic.

Proof. The result follows from the fact that D_θ is a vector bundle epimorphism. □

Lemma 2.3. *Let $K \subset M$ be a polyhedron of codim ≥ 1 and ω be a given 2-form. Then there exists an arbitrary C^0-small diffeotopy $h^\tau : M \to M$ such that ω can be C^0-approximated near $\tilde{K} = h^1(K)$ by a d_θ-exact 2-form $\tilde{\omega} = d_\theta(\tilde{a})$, where θ is a given closed 1-form.*

Proof. Given a 2-form ω we take F_ω as described in the above corollary.By an application of Holonomic Approximation Theorem [4], there exists a C^0-small diffeotopy $h^\tau : M \to M$ and a 1-form $\tilde{a}$ on M such that $j^1_{\tilde{a}}$ is C^0-close to F_ω along $\tilde{K} = h^1(K) \subset M$. Since D_θ is continuous, $D_\theta(j^1\tilde{a}) = d_\theta\tilde{a}$ must be arbitrarily close on $\tilde{K}$ to $D_\theta(F_\omega) = \omega$. Setting $\tilde{\omega} = d_\theta(\tilde{a})$ we get the desired result. □

Lemma 2.4. *Let K be a polyhedron in M of codim ≥ 1, ω be a 2-form on M and $a \in H^2_\theta(M;\mathbb{R})$ be a fixed cohomology class. Then there exists an arbitrary C^0-small diffeotopy $h^\tau : M \to M$, $\tau \in [0,1]$ such that ω can be C^0-approximated near $\tilde{K} = h^1(K)$ by a d_θ-closed 2-form $\tilde{\omega} \in a$, where θ is a given closed 1-form.*

Proof. We take an $\Omega \in a \in H^2_\theta(M;\mathbb{R})$ and set $\phi = \omega - \Omega$. Applying Lemma 2.4 to ϕ we get a Φ which is d_θ-exact. Now take $\tilde{\omega} = \Phi + \Omega$. Since Φ is d_θ-exact, we get $[\tilde{\omega}]_\theta = [\Omega]_\theta$. □

Proposition 2.5. *Let M be a smooth manifold and ω be a non-degenerate 2-form on M. Given any deRham cohomology class $\xi \in H^1(M)$, ω can be homotoped through non-degenrate 2-forms to a locally conformal symplectic form $d_\theta\alpha$, where the deRham cohomology class of θ is ξ.*

Proof. The proposition is a direct consequence of Theorem 1.3 which we prove below. □

Proof of Theorem 1.3. To prove the result we proceed as in [5]. Consider the canonical Grassmann bundle $G_{2n}(TM) \xrightarrow{\pi} M$ for which the fibres $\pi^{-1}(x)$ over a point $x \in M$ is the Grassmanian of $2n$-planes in T_xM. The space $Dist_q(M)$ of codimension q distributions on M can be identified with the section space $\Gamma(G_{2n}(M))$. We topologize $Dist_q(M)$ and $\Gamma(\wedge^2T^*M)$ with the compact open topology. Let Δ_q denote the subset of of $Dist_q(M) \times \Gamma(\wedge^2T^*M)$ defined by

$$\begin{array}{rcl} \Delta_q & := & \{(D,\omega) : (\iota_D^*\omega)^n \neq 0\} \text{ and} \\ \Phi_q & := & Fol_q(M) \times \Gamma(\wedge^2T^*M) \subset \Delta_q \end{array}$$

where $Fol_q(M) \subset Dist_q(M)$ is the space of all codimension q foliations on M. Therefore $(\mathcal{F}_0, \omega_0) \in \Phi_q$. Since non-degeneracy is an open condition the space of all $\mathcal{F}_0$-leafwise non-degenerate 2-forms on M is an open subset of $\Gamma(M,\wedge^2T^*M)$ in the fine C^0 compact-open topology. Therefore, there exists a continuous function $\epsilon : M \to \mathbb{R}_+$ such that if $\omega \in \Gamma(M,\wedge^2M)$ satisfies $dist(\omega(y),\omega_0(y)) < \epsilon(y)$, for all $y \in M$ then ω is $\mathcal{F}_0$-leafwise non-degenerate. Given $\theta \in \xi$ we can choose $\rho > 0$ and a d_θ-closed form ϕ which is ϵ-close to ω_0 on the ρ-nbd U_ρ of a core A of M (see Lemma 2.3). Next we take a smooth map $\chi : M \to [0,1]$ satisfying

$$\begin{array}{rcl} \chi(x) & \equiv & 0, \text{ for } x \notin U_\rho \\ \chi(x) & \equiv & 1, \text{ for } x \in U_{\rho/2} \end{array}$$

and define a homotopy of 2-forms on $[0,1/2]$ by

$$\begin{array}{rcl} \omega : [0,\frac{1}{2}] & \to & \Gamma(M,\wedge^2T^*M) \\ t & \mapsto & \omega_0 + 2t\chi(\phi - \omega_0) \end{array}$$

Then

(1) $\omega(0) = \omega_0$.
(2) $\omega(t)$ is $\mathcal{F}_0$-leafwise non-degenerate for all $t \in [0,1]$ since it is ϵ-close to ω_0.
(3) $\omega(\frac{1}{2}) = \phi$ is d_θ-closed on $U_{\frac{\rho}{2}}$.

Therefore if for $t \in [0, \frac{1}{2}]$ we set $\mathcal{F}(t) = \mathcal{F}_0$, then $(\mathcal{F}(t), \omega(t)) \in \Phi_q$.

Since M is an open manifold there exists an isotopy g_t, $0 \le t \le 1$, with $g_0 = id_M$ such that g_1 takes M into $U_{\rho/2}$. Now we define $(\mathcal{F}(t), \omega(t)) \in \Phi_q$ for $t \in [\frac{1}{2}, 1]$ by setting

$$\mathcal{F}(t) = g_{2t-1}^{-1}\mathcal{F}(\tfrac{1}{2}), \quad \omega(t) = g_{2t-1}^{*}\omega(\tfrac{1}{2}).$$

Therefore, $\omega(1)$ is $\mathcal{F}_1$-leafwise non-degenerate. Further, it is easy to see that $\omega(1)$ is $d_{g_1^*\theta}$ closed.

$$\begin{aligned}
d_{g_1^*\theta}\omega(1) &= d_{g_1^*\theta}(g_1^*\omega(\tfrac{1}{2})) \\
&= dg_1^*\omega(\tfrac{1}{2}) - g_1^*\theta \wedge g_1^*\omega(\tfrac{1}{2}) \\
&= g_1^*[d\omega(\tfrac{1}{2}) - \theta \wedge \omega(\tfrac{1}{2})] \\
&= g_1^* d_\theta \omega(\tfrac{1}{2}) = 0 \text{ (by (3) above)}
\end{aligned}$$

Since g_1 is homotopic to the identity map $[g_1^*\theta] = [\theta] = \xi$. Hence, $\omega(1)$ is a leafwise locally conformal symplectic form with leafwise Lee class defined by ξ. □

Remark 2.6. Note that $(\mathcal{F}_1, \omega_1)$ defines a leafwise locally conformal symplectic structure with Lee class $\xi_{\mathcal{F}}$.

Remark If we take θ to be the zero form we get back the result of Fernandes and Frejlich in [5].

3. The foliated contact case

In this section we prove h-principle for foliated contact manifold where the foliation under consideration is regular. The method is very similar to the l.c.s case.

Lemma 3.1. *Let M^n be a smooth manifold and $E = \wedge^1 M$. Then there exists a commutative diagram*

$$\begin{array}{ccc}
E^{(1)} & \xrightarrow{D} & \wedge^1 M \oplus \wedge^2 M \\
\downarrow & \circlearrowright & \downarrow \\
M & \xrightarrow{id_M} & M
\end{array}$$

where D is a vector bundle epimorphism such that the fibres of D are contractible. Moreover $D(j^1_{x_0}\alpha) = (\alpha, d\alpha)_{x_0}$.

Proof. Let $(x^1, ..., x^n)$ be the local coordinates around $x_0 \in M$ and $\alpha = \Sigma_{i=1}^n \alpha_i dx^i$ be the representation of α in this coordinates. Therefore,

$$j^1_{x_0}\alpha = (dx^1, ..., dx^n)(a_1, ..., a_n)^T + (dx^1, ..., dx^n)A(x^1 - x^1_0, ..., x^n - x^n_0)^T$$

where $a_i = \alpha_i(x_0)$, $A = (a_{ij})_{n\times n} = (\frac{\partial \alpha_i}{\partial x^j}(x_0))$. Hence, $j^1_{x_0}(\alpha) = (a_i, a_{ij}) \in \mathbb{R}^{n+n^2}$. Define D by

$$D(j^1_{x_0}\alpha) = (\Sigma_{i=1}^n a_i dx^i, \Sigma_{i<j}(a_{ij} - a_{ji})dx^i \wedge dx^j) = (\alpha(x_0), d\alpha(x_0))$$

Then D is clearly an epimorphism since the following system of equations

$$a_i = b_i \quad \text{and} \quad a_{ij} - a_{ji} = b_{ij} \quad \text{for all } i \neq j,\ i,j = 1, ..., n$$

is clearly solvable in a_i and a_{ij}. Hence the fibres of D are affine subspaces. □

Remark : Any section $(\theta,\omega) : M \to \wedge^1 M \oplus \wedge^2 M$ can be lifted up to a section $F_{(\theta,\omega)} : M \to E^{(1)}$ such that $D \circ F_{(\theta,\omega)} = (\theta,\omega)$ and any two such lifts of a given (θ,ω) are homotopic.

Lemma 3.2. *Let $K \subset M$ be a polyhedron of codim ≥ 1 and (θ,ω) be an element of $\Gamma(M, \wedge^1 M \oplus \wedge^2 M)$. Then there exists an arbitrary C^0-small diffeotopy $h^\tau : M \to M$ such that (θ,ω) C^0-approximated near $\tilde{K} = h^1(K)$ by $(\tilde{\theta}, d\tilde{\theta})$ for some 1-form $\tilde{\theta}$.*

Proof. Take $F_{(\theta,\omega)}$ for the given (θ,ω) and choose a holonomic approximation [4] $j^1_{\tilde{\theta}}$ along $\tilde{K} = h^1(K) \subset M$, where h^τ is an arbitrary C^0-small diffeotopy. Now extend $\tilde{\theta}$ to all of M. Since $j^1\tilde{\theta}$ is C^0-close to $F_{(\theta,\omega)}$ on $\tilde{K}$ and D is continuous, $D(j^1\tilde{\theta}) = (\tilde{\theta}, d\tilde{\theta})$ is C^0-close to $DF_{(\theta,\omega)} = (\theta,\omega)$ on $\tilde{K}$. □

Proof of Theorem 1.4. Let D_0 denote the distribution defined by the foliation $\mathcal{F}_0$. Define Δ_q, Φ_q as subsets of $Dist_q(M) \times \Omega^1(M) \times \Omega^2(M)$ as follows:

$$\begin{aligned} \Delta_q &:= \{(D,\theta,\omega) : (\iota_D^*\theta) \wedge (\iota_D^*\omega)^n \neq 0 \text{ at each point of } M\} \\ \Phi_q &:= Fol_q(M) \times \Gamma(M, \wedge^1 M \oplus \wedge^2 M) \subset \Delta_q, \end{aligned}$$

where i_D is the inclusion of D in TM. By the given hypothesis, $(\mathcal{F}_0, \theta_0, \omega_0)$ is in Φ_q. Since $\iota_{D_0}^*\theta \wedge (\iota_{D_0}^*\omega)^n \neq 0$ defines an open differential relation for pairs (θ,ω), we get a positive function $\varepsilon : M \to \mathbb{R}$ such that $\theta \wedge \omega^n$ is nowhere vanshing on the leaves of $\mathcal{F}_0$ whenever $dist\{(\theta(y),\omega(y)), (\theta_0(y),\omega_0(y))\} < \varepsilon(y)$, for all $y \in M$; in other words, if (θ,ω) lies in the ε-neighbourhood of (θ_0,ω_0), the triple $(D_0,\theta,\omega) \in \Delta_q$. By Lemma 3.2 above we can choose $\rho > 0$ and a one form ϕ such that $(\phi, d\phi)$ is ϵ-close to (θ_0,ω_0) on the ρ neighbourhood U_ρ of a core A in M.

Next we take a smooth map $\chi : M \to [0,1]$ satisfying

$$\begin{aligned} \chi(x) &\equiv 0, \text{ for } x \notin U_\rho \\ \chi(x) &\equiv 1, \text{ for } x \in U_{\rho/2} \end{aligned}$$

Define a continuous map

$$\begin{aligned} (\theta,\omega) : [0,1/2] &\to \Gamma(M, \wedge^1 M \oplus \wedge^2 M) \\ t &\mapsto (\theta_0,\omega_0) + 2t\{(\phi, d\phi) - (\theta_0,\omega_0)\}. \end{aligned}$$

It may be seen easily that (θ,ω) satisfies the following:

(1) $(\theta,\omega)(0) = (\theta_0,\omega_0)$.
(2) $\theta(t) \wedge [\omega(t)]^n \neq 0$ on the leaves of $\mathcal{F}_0$.
(3) $d\theta(1/2) = \omega(1/2)$ on $U_{\rho/2}$.

For $t \in [0,1/2]$ we set $\mathcal{F}(t)$ to be the stationary homotopy at $\mathcal{F}_0$, so that $t \mapsto (\mathcal{F}(t), \theta(t), \omega(t))$ takes values in Φ_q. For the second half of the homotopy we can choose an isotopy $g_t : M \to M$ between $g_0 = id_M$ and $g_1(M) \hookrightarrow U_{\rho/2}$. Such an isotopy exists because U_ρ is a neighbourhood of a core of M. Now we define $(\mathcal{F}(t), \theta(t), \omega(t)) \in \Phi_q$, $t \in [1/2, 1]$ by setting

$$\mathcal{F}(t) = (g_{2t-1})^*(\mathcal{F}(\tfrac{1}{2})), \quad \theta(t) = (g_{2t-1})^*(\theta(\tfrac{1}{2})), \quad \omega(t) = (g_{2t-1})^*(\omega(\tfrac{1}{2})).$$

Finally observe that

$$d\theta(1) = d[g_1^*(\theta(1/2))] = g_1^*[d\theta(1/2)] = g_1^*[\omega(1/2)] = \omega(1).$$

Therefore, $\theta(1)$ is a $\mathcal{F}_1$-leafwise contact form. □

Remark 3.3. The integrability condition on the initial distribution in Theorems 1.3 and 1.4 can be relaxed to the extent that it is enough to have it homotopic to the distribution of a foliation. This can be seen by taking into account the classification of foliations due to Haeflier [8]. We refer to [5] for a detailed argument.

Remark 3.4. Finally we remark that an analogue of Theorem 1.1 also holds true in the contact case. Suppose that M is a smooth manifold with a foliation $\mathcal{F}$ of dimension $2n+1$. Let (α, β) be a pair consisting of a foliated 1-form α and a foliated 2-form β on $(M, \mathcal{F})$ such that $\alpha \wedge \beta^n$ is nowhere vanishing on the leaves of $\mathcal{F}$. The leafwise non-vanishing condition on (α, β) is an open condition and hence defines an open subset $\mathcal{R}$ in the 1-jet space $E^{(1)}$, where $E \to M$ is the vector bundle whose total space is $T^*\mathcal{F} \oplus \Lambda^2(T^*\mathcal{F})$. Since the condition is invariant under the action of foliated diffeotopies Bertelson's theorem applies to this relation provided we assume that the foliation $(M, \mathcal{F})$ satisfies certain openness condition as defined in [1]. Hence, the given pair (α, β) can be homotoped through such pairs to $(\eta, d_{\mathcal{F}}\eta)$ for some foliated 1-form η on $(M, \mathcal{F})$, so that η is a foliated contact form. (Here $d_{\mathcal{F}}$ denotes the coboundary map of the foliated deRham complex).

4. Regular Jacobi structures on open manifolds

We can reformulate Theorem 1.4 in terms of the Jacobi structure as follows. Let $\nu^k(M)$ denote the space of sections of the alternating bundle $\Lambda^k(TM)$. We shall refer to these sections as k-multivector fields on M. A pair $(\Lambda, E) \in \nu^2(M) \times \nu^1(M)$ will be called a regular pair if $\mathcal{D} = \Lambda^{\#}(T^*M) + \langle E \rangle$ is a regular distribution on M.

Theorem 4.1. *Let $(\Lambda_0, E_0) \in \nu^2(M) \times \nu^1(M)$ be a regular pair an open manifold M. Suppose that the distribution $\mathcal{D}_0 := \Lambda_0^{\#}(T^*M) + \langle E_0 \rangle$ is odd dimensional. Then (Λ_0, E_0) can be homotoped through regular pairs to a Jacobi pair (Λ_1, E_1) provided the distribution $\mathcal{D}_0$ is homotopic to an integrable distribution.*

Proof. Let (Λ, E) be a regular pair and the distribution $\mathcal{D} = \Lambda^{\#}(T^*M) + \langle E \rangle$ is odd dimensional. Define a section α of $\mathcal{D}^*$ by the relations

$$\alpha(\mathrm{Im}\,\Lambda^{\#}) = 0 \quad \text{and} \quad \alpha(E) = 1. \tag{1}$$

Define β as a section of $\wedge^2(\mathcal{D}^*)$ by

$$\beta(\Lambda^{\#}\eta, \Lambda^{\#}\xi) = \Lambda(\eta, \xi), \qquad i_E\beta = 0, \tag{2}$$

where $i(E)$ denotes the interior multiplication by E. It can be shown that $\alpha \wedge \beta^n$ is nowhere vanishing. On the other hand given a pair (α, β) as above we can define a vector bundle isomorphism $\phi : \mathcal{D} \to \mathcal{D}^*$ by

$$\phi(X) = i_X(\beta) + \alpha(X)\alpha.$$

Let

$$E = \phi^{-1}\alpha \tag{3}$$

and let Λ be so defined that the following diagram commutes:

$$
\begin{array}{ccc}
T^*M & \stackrel{\Lambda^\#}{\longrightarrow} & TM \\
i^*\downarrow & \circlearrowleft & \uparrow i \\
\mathcal{D}^* & \stackrel{\phi^{-1}}{\longrightarrow} & \mathcal{D}
\end{array}
\tag{4}
$$

Then (Λ, E) is a regular Jacobi pair. Thus there is a one to one correspopodence between regular pairs (Λ, E) and the triples $(\mathcal{D}, \alpha, \beta)$ such that $\alpha \wedge \beta^n$ is nowhere vanishing. Further, the regular contact foliations correspond to regular Jacobi pairs with odd-dimensional characteristic distributions under this correspondence.

The result now follows directly from Theorem 1.4. Let (Λ_0, E_0) be as in the hypothesis and $\mathcal{F}_0$ be the foliation associated with $\mathcal{D}_0$. Then we can define (α_0, β_0) by the equations (1) and (2) so that $\alpha_0 \wedge \beta_0^n$ is non-vanishing on $\mathcal{D}_0$. By Theorem 1.4, we obtain a homotopy $(\mathcal{F}_t, \alpha_t, \beta_t)$ of $(\mathcal{F}_0, \alpha_0, \beta_0)$ such that $\alpha_t \wedge \beta_t^n$ is a nowhere vanishing form on $T\mathcal{F}_t$ and $\beta_1 = d\alpha_1$, so that α_1 is a $\mathcal{F}_1$-leafwise contact form. Let (Λ_t, E_t) be defined by (3) and the diagram (4) □

5. Appendix 1: Preliminaries of Jacobi manifolds

A Jacobi structure on a smooth manifold M is given by a pairing (Λ, E), where Λ is a bivector field i.e $\Lambda \in \Gamma(\wedge^2 TM)$ and E is a vector field on M, satisfying

$$[\Lambda, \Lambda] = 2E \wedge \Lambda, \qquad [E, \Lambda] = 0, \tag{5}$$

where [,] is the Schouten bracket. If $E = 0$ then Λ is a Poisson structure on M.

Example 5.1. Let M be a $2n$-dimensional manifold with a symplectic form (i.e. a closed, non-degenerate 2-form) ω. The non-degeneracy condition implies that $b : \Gamma(TM) \to \Gamma(T^*M)$, given by $b(X) = \omega(X, -)$, is a vector bundle isomorphism. Then M has a Poisson structure defined by

$$\pi(\alpha, \beta) = \omega(b^{-1}(\alpha), b^{-1}(\beta)), \quad \text{for all } \alpha, \beta \in T_x^*M, x \in M.$$

Example 5.2. A locally conformal symplectic manifold (in short, l.c.s manifold) is an even dimensional manifold M with a non-degenerate 2-form ω and a closed 1-form θ such that

$$d\omega = \theta \wedge \omega.$$

This equation can be expressed as $d_\theta(\omega) = 0$, where $d_\theta = d - \theta\wedge$. It can be seen easilty that $d_\theta^2 = 0$; hence d_θ is a coboundary operator. The 1-form θ is called the Lee form of the l.c.s structure ω and the deRham cohomology class of θ is called the Lee class of ω. For an l.c.s manifold (M, ω, θ) the Jacobi pairing is given by

$$\Lambda(\alpha, \beta) = \omega(b^{-1}(\alpha), b^{-1}(\beta)) \quad \text{and} \quad E = b^{-1}(\theta),$$

where $b : \Gamma(TM) \to \Gamma(T^*M)$ is the isomorphism given by $b(X) = \omega(X, -)$.

Example 5.3. Contact manifolds are the odd dimensional counterpart of symplectic manifolds. A 1-form η on a $(2n + 1)$-dimensional manifold is said to be a contact form on M if $\eta \wedge (d\eta)^n$ is nowhere vanishing. For contact manifold (M, η) the Jacobi pairing is given by

$$\Lambda(\alpha, \beta) = d\eta(b^{-1}(\alpha), b^{-1}(\beta)), \quad \text{and} \quad E = b^{-1}(\eta),$$

where $b : \Gamma(TM) \to \Gamma(T^*M)$ is the isomorphism given by $b(X) = d\eta(X, -) + \eta(X)\eta$.

Let (M,Λ,E) be a Jacobi manifold. The bivector field Λ defines a bundle homomorphism $\Lambda^{\#}: T^*M \to TM$ by

$$\Lambda^{\#}(\alpha)(\beta) = \Lambda(\alpha,\beta),$$

where $\alpha,\beta \in T^*_xM$, $x \in M$. The Jacobi pair (Λ, E) defines a distribution $\mathcal{D}$ as follows:

$$\mathcal{D}(x) = \Lambda^{\#}(T^*_xM) + \langle E_x\rangle, x \in M$$

$\mathcal{D}$ is, in general, a singular distribution; however, it integrates to a (singular) foliation $\mathcal{F}$ on M. This foliation is referred as the characteristic foliation of the Jacobi manifold. If $\mathcal{D} = TM$ then the manifold is locally conformally symplectic or contact according as the dimension of M is even or odd. More generally, the leaves of the characteristic foliation are either locally conformally symplectic or contact with the induced Jacobi structures on them. In particular, when $E = 0$, Λ is a Poisson bivector field and induced Poisson structure on the leaves are non-degenerate. Thus the leaves are symplectic manifolds.

A Jacobi manifold will be called *regular* if its characteristic foliation is regular. Thus a regular Jacobi manifold has a regular l.c.s foliation or a regular contact foliation depending on the dimension of the foliation. On the other hand given a regular foliation with leafwise l.c.s or contact form there is a Jacobi structure on M whose characteristic foliation is the given one. By an l.c.s. structure on a regular foliation $\mathcal{F}$, we mean a pair (ω,θ), where ω is leafwise non-degenerate foliated 2-form and θ is a foliated closed 1-form such that $d_\theta\omega = 0$ in the foliated deRham cohomology. The cohomology class of θ in $H^2(M,\mathcal{F})$ is called the Lee class of the foliated l.c.s. structure ω.

For general theory of symplectic and contact manifolds and Poisson, Jacobi manifolds we refer to [2], [9] and [10].

6. Appendix 2: Holonomic Approximation Theorem

Let $p : E \to M$ be a smooth fibration and $E^{(r)} \to M$ be the r-jet bundle associated to the sections of E. A subset $\mathcal{R}$ of $E^{(r)}$ is said to be an r-th order *partial differential relation* (or simply a *relation*). A section f of E is said to be a *solution* of $\mathcal{R}$ if the image of its r-jet map j^r_f is contained in $\mathcal{R}$. A section of $E^{(r)}$ is said to be *holonomic* if it is the r-jet of some section of E.

Let A be a polyhedron (possibly non-compact) in M of positive codimension. Let σ be any section of the r-jet bundle $E^{(r)}$ over $Op\,A$. The Holonomic Approximation Theorem (see [4]) says that given any positive functions ε and δ on M there exist a (small) diffeotopy δ_t and a holonomic section $\sigma' : Op\,\delta_1(A) \to E^{(1)}$ such that

(1) $dist(x, \delta_t(x)) < \delta(x)$ for all $x \in M$ and $t \in [0,1]$ and
(2) $dist(\sigma(x), \sigma'(x)) < \varepsilon(x)$ for all $x \in Op\,(\delta_1(A))$.

This means that under the given condition any f can be approximated (in the fine C^0 topology) by a holonomic section in a neighbourhood of $\delta_1(A)$, where δ_t is an arbitrary small diffeotopy of M.

Remark 6.1. By a small diffeotopy we mean that $\delta_t(A)$ remains within the domain of f for all t.

We can summarise the main argument used in the proofs of Theorems 1.3 and 1.4 in the following result.

Theorem 6.2. *Let M be an open manifold and $\mathcal{R}$ an open differential relation for sections of a vector bundle $E \to M$. Then given any section σ of $\mathcal{R}$ there exist a core K of M and a holonomic section $\sigma' : Op\,K \to E$ such that the linear homotopy between σ and σ' lies completely within $\Gamma(\mathcal{R})$ over $Op\,K$.*

Proof. Since $\mathcal{R}$ is an open subset of $E^{(r)}$, the space of sections of $\mathcal{R}$ is an open subset of $\Gamma(E^{(r)})$ in the fine C^0-topology. Therefore, given a section σ of $\mathcal{R}$, there exists a positive function ε satisfying the following condition:

$$\sigma' \in \Gamma(E^{(r)}) \text{ and } \mathrm{dist}\,(\sigma(x), \sigma'(x)) < \varepsilon(x) \quad \Rightarrow \sigma' \text{ is a section of } \mathcal{R}$$

Consider a core A of M and a δ-tubular neighbourhood of A for some positive δ. By the Holonomic Approximation Theorem there exist a diffeotopy δ_t and a holonomic section σ' satisfying (1) and (2). First note that $K = \delta_1(A)$ is again a core of M. Moreover, since σ' lies in the ε-neighbourhood of σ, the linear homotopy between σ and σ' lies completely within $\mathcal{R}$ by the the choice of ε. This completes the proof of the proposition. □

Remark 6.3. The core K can not be fixed apriori in the statement of the proposition.

Secondly, if we donot require the homotopy between σ and σ' to be linear then we do not need to assume that E is a vector bundle.

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